

DENSITY CORRELATORS IN A SELF-SIMILAR CASCADE

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Abstract

Multivariate density moments (correlators) of arbitrary order are obtained for the multiplicative self-similar cascade. This result is based on the calculation by Greiner, Eggers and Lipa (reference [1]) where the correlators of the *logarithms* of the particle densities have been obtained. The density correlators, more suitable for comparison with multiparticle data, appear to have even simpler form than those obtained in [1].

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1 Introduction

It has been shown recently [1, 2, 3] that one can express the multivariate generating function for a random multiplicative cascade process in a simple analytical form. Using such a generating function one can calculate multivariate density moments (correlators) [4] of any order and for an arbitrary number of bins. It was also argued in [1, 2, 3] that the cumulant density correlators obtained in this way take a particularly simple form. They can be expressed by derivatives of the generating function describing one vertex, i.e. a single-step cascade. This beautiful result, however, required the generating function and the correlators to be expressed in terms of logarithm of the density rather than in terms of the density itself. Although mathematically elegant and adequate, e.g., for studies of turbulence, such representation is difficult to use in analysis of multiparticle production, as it makes the comparison with the data somewhat complicated. The moments expressed directly in terms of the density have been also discussed in [5] but only the recursive formula describing them was given there.

Indeed, the experimental determination of the moments of particle density is greatly simplified by the observation that they can be approximated by factorial moments of the measured particle distribution [6]. As this procedure is not applicable for the moments of the logarithm of density, the corresponding measurements must involve much more sophisticated methods. This difficulty is particularly acute for low multiplicities where Poisson fluctuations of particle number can substantially affect the estimate of the logarithm of density.

In the present note we point out that the generating function derived in [1, 2, 3] can also serve for calculation of the standard density moments and correlators and thus can be used directly for analysis of multiparticle production. The obtained analytic expressions are rather simple: they do not even require differentiation but simply evaluation of the generating function at integer values of its parameters. They can be considered as a generalization to arbitrary cascades and arbitrary correlators of the results obtained in [4] for a particular case of the α -model. Although our formulae represent a simple reformulation of those derived in [1, 2, 3], we feel that they deserve attention because, as explained above, they are better suited for analysis of multiparticle data, particularly at moderate multiplicities.

In the next section we derive directly the formula for multivariate density moments (correlators) in a multiplicative cascade. In Section 3 we discuss the examples of one, two and three-point correlators. Our conclusions and comments are listed in the last section.

2 Multi-bin correlators

We shall use the notation of Ref. [1]. The evolution of the self-similar cascade goes as follows. One starts with one bin of a given width Δ , which is characterised by some quantity ϵ , which can be

the energy dissipation density, the density of particles etc., depending on the particular process to be described by the cascade model. Its value can be chosen to be $\epsilon = 1$ without losing generality.

This bin, the mother interval, is split into two bins², the daughter intervals. The contents of the two daughter bins is obtained by multiplying that of the mother interval, $\epsilon = 1$, by two numbers q_0 and q_1 drawn from the probability distribution $p(q_0, q_1)$ which is often called splitting function and which we consider to be symmetric, $p(q_0, q_1) = p(q_1, q_0)$. The resulting densities are

$$\begin{aligned}\epsilon_0 &= q_0 \epsilon = q_0 \\ \epsilon_1 &= q_1 \epsilon = q_1.\end{aligned}\tag{1}$$

Now the two bins become themselves the mother intervals and are subsequently split into two, giving after the second step the densities

$$\begin{aligned}\epsilon_{00} &= q_{00} q_0 & \epsilon_{01} &= q_{01} q_0 \\ \epsilon_{10} &= q_{10} q_1 & \epsilon_{11} &= q_{11} q_1,\end{aligned}\tag{2}$$

where the pairs of multipliers (q_{00}, q_{01}) and (q_{10}, q_{11}) are again drawn independently from the probability distribution p , the same which was used at the previous step of the cascade.

After J steps, one obtains 2^J bins, each one addressed by the binary index $k_1 k_2 \dots k_J$, where every k_j equals 0 or 1. The density of every bin is the product of J random multipliers q which follow the path leading from the original mother interval to this particular bin:

$$\epsilon_{k_1 k_2 \dots k_J} = \prod_{j=1}^J q_{k_1 k_2 \dots k_j}.\tag{3}$$

Now, the probability distribution of the particular configuration of the densities $\epsilon_{k_1 \dots k_J}$ is a product of the splitting functions $p(q_{k_1 \dots k_{j-1} 0}, q_{k_1 \dots k_{j-1} 1})$ taken at each branching point of the cascade, convoluted with the appropriate δ functions:

$$p(\epsilon_{0\dots 0}, \dots, \epsilon_{1\dots 1}) = \int \left[\prod_{j=1}^J \prod_{k_1, \dots, k_{j-1}=0}^1 dq_{k_1 \dots k_{j-1} 0} dq_{k_1 \dots k_{j-1} 1} p(q_{k_1 \dots k_{j-1} 0}, q_{k_1 \dots k_{j-1} 1}) \right] \left[\prod_{k_1, \dots, k_J=0}^1 \delta \left(\epsilon_{k_1 \dots k_J} - \prod_{j=1}^J q_{k_1 \dots k_j} \right) \right].\tag{4}$$

The moments of the logarithms of the densities $\epsilon_{k_1 \dots k_J}$ can be calculated from the generating function Z_T defined in [3]:

$$Z_T(\sigma_{0\dots 0}, \dots, \sigma_{1\dots 1}) = \left\langle \exp \left(\sum_{k_1, \dots, k_J=0}^1 \sigma_{k_1 \dots k_J} \ln \epsilon_{k_1 \dots k_J} \right) \right\rangle,\tag{5}$$

² This is the case of the binary cascade. Generalization of all the results obtained here to a cascade characterized by three- or more-fold splitting is straightforward

where the averaging goes over all configurations of the densities occuring with the probability (4). Z_T provides, by differentiation, the moments K of the logarithms of the densities ϵ :

$$K(t_{0\dots 0}, \dots, t_{1\dots 1}) = \left\langle \prod_{k_1, \dots, k_J=0}^1 ([\ln \epsilon_{k_1 \dots k_J}])^{t_{k_1 \dots k_J}} \right\rangle = \frac{\partial^{t_{0\dots 0}}}{\partial (\sigma_{0\dots 0})^{t_{0\dots 0}}} \cdots \frac{\partial^{t_{1\dots 1}}}{\partial (\sigma_{1\dots 1})^{t_{1\dots 1}}} Z_T(\sigma_{0\dots 0}, \dots, \sigma_{1\dots 1}) \Big|_{\sigma_{k_1 \dots k_J}=0}. \quad (6)$$

It was shown also in [2, 3] that Z_T factorizes:

$$Z_T(\sigma_{0\dots 0}, \dots, \sigma_{1\dots 1}) = \prod_{j=1}^J \prod_{k_1, \dots, k_{j-1}=0}^1 Z_T(\sigma_{k_1 \dots k_{j-1} 0}, \sigma_{k_1 \dots k_{j-1} 1}), \quad (7)$$

where

$$Z_T(\sigma_0, \sigma_1) = \int dq_0 dq_1 p(q_0, q_1) \exp(\sigma_0 \ln q_0 + \sigma_1 \ln q_1) \quad (8)$$

is the binary generating function corresponding to a cascade consisting of only one step and

$$\sigma_{k_1 \dots k_j} = \sum_{l=j+1}^J \sum_{k_{j+1}, \dots, k_l=0}^1 \sigma_{k_1 \dots k_l} \quad (9)$$

is the sum of all σ which are descendant with respect to $\sigma_{k_1 \dots k_j}$. For example, $\sigma_0 = \sigma_{00} + \sigma_{01}$, where $\sigma_{00} = \sigma_{000} + \sigma_{001}$ and $\sigma_{01} = \sigma_{010} + \sigma_{011}$, and so on till the last step of the cascade.

Let us now consider the multivariate moments of the densities $\epsilon_{k_1 \dots k_J}$ themselves:

$$M(s_{0\dots 0}, \dots, s_{1\dots 1}) = \left\langle \prod_{k_1, \dots, k_J=0}^1 (\epsilon_{k_1 \dots k_J})^{s_{k_1 \dots k_J}} \right\rangle, \quad (10)$$

where $s_{k_1 \dots k_J}$ can be any non-negative integer numbers³. The average is, again, taken over all configurations at the end of the cascade:

$$M(s_{0\dots 0}, \dots, s_{1\dots 1}) = \int \left[\prod_{k_1, \dots, k_J=0}^1 d\epsilon_{k_1 \dots k_J} (\epsilon_{k_1 \dots k_J})^{s_{k_1 \dots k_J}} \right] p(\epsilon_{0\dots 0}, \dots, \epsilon_{1\dots 1}), \quad (11)$$

Comparing (10) with (5) one notices that taking the generating function Z_T , (5), at fixed integer points $s_{0\dots 0}, \dots, s_{1\dots 1}$ gives directly the moments M :

$$M(s_{0\dots 0}, \dots, s_{1\dots 1}) = Z_T(s_{0\dots 0}, \dots, s_{1\dots 1}). \quad (12)$$

The factorization (7) can thus also be reformulated in terms of the moments M :

$$M(s_{0\dots 0}, \dots, s_{1\dots 1}) = \prod_{j=1}^J \prod_{k_1, \dots, k_{j-1}=0}^1 M(s_{k_1 \dots k_{j-1} 0}, s_{k_1 \dots k_{j-1} 1}). \quad (13)$$

³ The formulae below are valid for any non-negative $s_{k_1 \dots k_J}$. Only for integer $s_{k_1 \dots k_J}$, however, the correlators M can be represented by the factorial correlators and thus easily estimated from data.

The rule (9) holds here similarly for the variables $s_{k_1 \dots k_j}$. The hierarchy of these variables sitting at each link and of the binary functions M corresponding to every branching point is shown in Fig. 1 on the example of a 3-step cascade. The binary moments in the r.h.s. of Eq (13) correspond to the binary generating function $Z_T(\sigma_0, \sigma_1)$ taken again at integer points $\sigma_0 = s_0$, $\sigma_1 = s_1$ and they read:

$$M(s_0, s_1) = \int dq_0 dq_1 (q_0)^{s_0} (q_1)^{s_1} p(q_0, q_1). \quad (14)$$

We thus conclude that all the density correlators can be expressed by products of the binary moments $M(s_0, s_1)$. This is surely a strong constraint which can be tested against the experimental data on factorial correlators [4]. The simplest possibility of such a test is discussed in the next section.

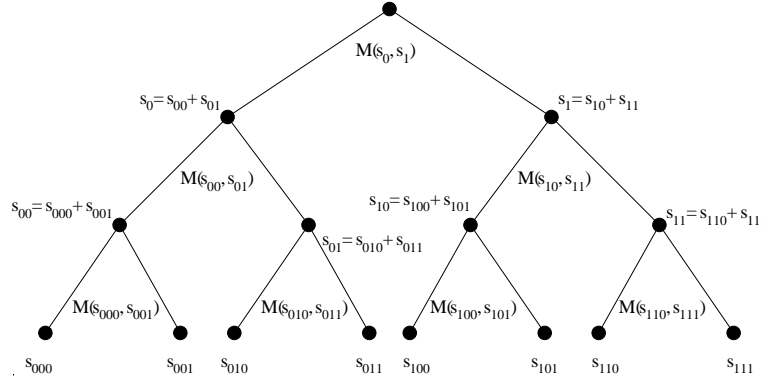


Figure 1: Hierarchy of the integer variables s and the functions M illustrating Eq. (13).

3 Examples

Consider first the t^{th} -order moment calculated for a single bin addressed by a binary number $\lambda = l_1 \dots l_J$:

$$M(0, 0, \dots, t, \dots, 0) = \langle (\epsilon_\lambda)^t \rangle. \quad (15)$$

This is the particular case of Eq. (13), where all the variables s_κ addressed by $\kappa = k_1 \dots k_J$ different from λ are set to zero, whereas $s_\lambda = t$. One can notice that $s_{l_1 \dots l_j} = t$ for any $j < J$ and $s_{k_1 \dots k_{J-j}} = 0$ for all other $k_1 \dots k_J \neq \lambda$. Thus, knowing that $M(0, 0) = 1$, one concludes that the moment $M(0, \dots, t, \dots, 0)$ is the product of J binary moments $M(t, 0)$ or $M(0, t)$ which are all equal to each other due to symmetry and sit at the vertices of the trajectory leading to the considered bin. The resulting moment reads

$$\langle (\epsilon_\lambda)^t \rangle = [M(t, 0)]^J \quad (16)$$

and it does not depend on the address λ of the particular bin. Here, we recover the well-known result, namely the scaling law of the density moments [6]. The graphic representation of the result (16) is shown in Fig. 2.

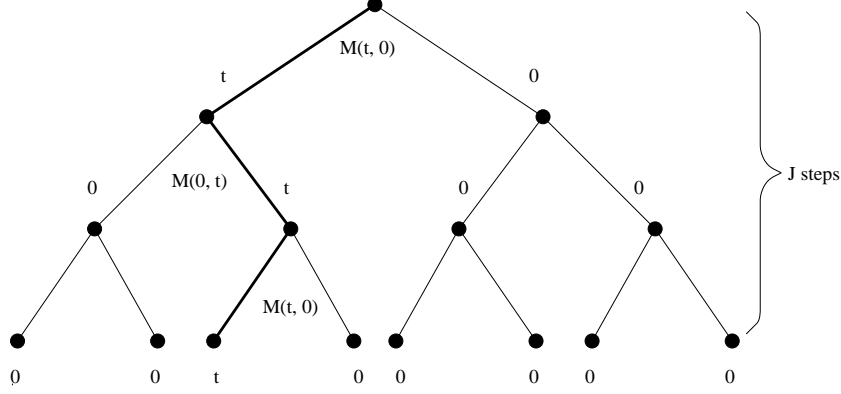


Figure 2: Graphic representation of the one-point moment of order t for $J = 3$. The moment reads $M(0, 0, t, 0, 0, 0, 0, 0) = \langle (\epsilon_{010})^t \rangle = [M(t, 0)]^3$.

Similarly, one can calculate the two-point correlators of order (t, u) between bins addressed by λ and μ ,

$$M(0, \dots, t, \dots, u, \dots, 0) = \langle (\epsilon_\lambda)^t (\epsilon_\mu)^u \rangle. \quad (17)$$

The graphic representation of this correlator is shown in Fig. 3 and the result is

$$\langle (\epsilon_\lambda)^t (\epsilon_\mu)^u \rangle = [M(t + u, 0)]^{J-d} M(t, u) [M(t, 0) M(u, 0)]^{d-1}, \quad (18)$$

where d is the ultrametric distance [2] between considered bins, i.e. the number of generations leading to their common ancestor vertex. In other words, the ultrametric distance between two bins addressed by the binary indices $\lambda = l_1 \dots l_J$ and $\mu = m_1 \dots m_J$, equals d if

$$l_j = m_j \quad \text{for} \quad j \leq J - d \quad (19)$$

and

$$l_{J-d+1} \neq m_{J-d+1}. \quad (20)$$

Calculating the 3-point correlator is now straightforward. Consider the correlator of the order (t, u, v) between three bins addressed by indices λ , μ and ν , respectively. It depends only on the trajectories leading to the bins. Let us assume that the ultrametric distance between the first two bins, $d_{\lambda\mu}$ is smaller than that of the second pair of the bins, $d_{\mu\nu}$. This assumption does not restrict the generality of the solution. For such ultrametric distances between the bins, the 3-point correlator reads:

$$\begin{aligned} \langle (\epsilon_\lambda)^t (\epsilon_\mu)^u (\epsilon_\nu)^v \rangle = & \\ & [M(t + u + v, 0)]^{J-d_{\mu\nu}} M(t + u, v) [M(v, 0)]^{d_{\mu\nu}-1} [M(t + u, 0)]^{d_{\mu\nu}-d_{\lambda\mu}-1} \\ & \times M(t, u) [M(t, 0) M(u, 0)]^{d_{\lambda\mu}-1}. \end{aligned} \quad (21)$$

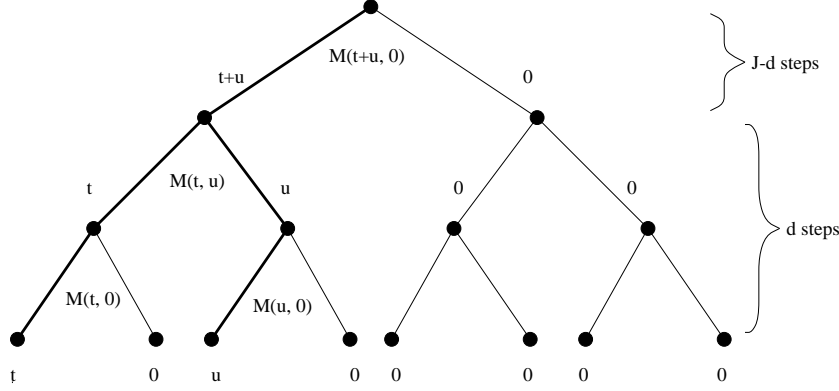


Figure 3: Graphic representation of the two-point correlator of order (t, u) for $J = 3$ and $d = 2$. The correlator is $M(t, 0, u, 0, 0, 0, 0) = \langle (\epsilon_{000})^t (\epsilon_{010})^u \rangle = M(t + u, 0)M(t, u)M(t, 0)M(u, 0)$.

Eqs (16), (18) and (21) provide, at least in principle, a powerful test of the cascade model. Indeed, (16) allows to determine the standard intermittency parameter from a linear fit

$$\ln \langle (\epsilon_\lambda)^t \rangle = f_t \ln \left(\frac{\Delta}{\delta} \right), \quad (22)$$

where

$$f_t = \frac{\ln M(t, 0)}{\ln 2}, \quad (23)$$

Δ is the total width of the interval in which the measurement is taken and δ is the width of the bins into which the interval Δ has been divided.

Similarly, from (18) one obtains the relation:

$$\frac{1}{\ln 2} \ln \left[\frac{\langle (\epsilon_\lambda)^t (\epsilon_\mu)^u \rangle}{\langle (\epsilon_\lambda)^t \rangle \langle (\epsilon_\mu)^u \rangle} \right] = (J - d)(f_{t+u} - f_t - f_u) + (f_{t,u} - f_t - f_u) \quad (24)$$

where

$$f_{t,u} = \frac{\ln M(t, u)}{\ln 2}. \quad (25)$$

We see that the r.h.s. of Eq. (24) depends only on $J - d$, i.e. on the number of steps of the cascade leading to the splitting point in which the trajectories of the bins λ and μ separate. It is thus independent of the bin width δ . This feature is very well confirmed by the data [7]. The reason for this is the fact that the above number of steps is not sensitive to the structure of the cascade splittings hidden within the bins of width δ (this holds provided that the cascade splittings conserve the particle density; in the case it does not, one can calculate the appropriate corrections [8]).

Another important feature is that the correlator (24) does not depend on the physical distance D between the bins for $d = J$. It means that for the binary cascade this correlator stays constant

for distances D larger than $\Delta/2$, its value being $f_{t,u} - f_t - f_u$ (For a cascade with tripple splitting it would remain constant for $D > 2/3\Delta$). It means that knowing the intermittency index f_s for arbitrary s and the above base-level value of the correlator allows one to determine $f_{t,u}$.

The dependence of the correlators on ultrametric distance d is also worth studying. This is not easy matter, however, because the relation of d to the distance D between the bins is not simple and, moreover, depends on the dimensionality of the problem⁴. Nevertheless, such an analysis seems not out of reach [10].

Finally, from Eq. (21) one obtains the relation:

$$\begin{aligned} \frac{1}{\ln 2} \ln \left[\frac{\langle (\epsilon_\lambda)^t (\epsilon_\mu)^u (\epsilon_\nu)^v \rangle}{\langle (\epsilon_\lambda)^t \rangle \langle (\epsilon_\mu)^u \rangle \langle (\epsilon_\nu)^v \rangle} \right] = \\ (J - d_{\mu\nu})(f_{t+u+v} - f_t - f_u - f_v) + (d_{\mu\nu} - d_{\lambda\mu})(f_{t+u} - f_t - f_u) + \\ (f_{t+u,v} - f_{t+u} - f_v) + (f_{t,u} - f_t - f_u). \end{aligned} \quad (26)$$

This relation can allow for checking the consistency of the above results since it does not contain any new parameters. As it was noted before, the coefficients f_s and $f_{s,t}$ can be determined for arbitrary s and t using the relations (22) and (24).

4 Conclusions

In conclusion, we have rewritten the results of Ref. [1, 2] in a form which seems better suited for application to data on multiple production. The resulting general formula for density correlators of any order can be directly compared to multiparticle data on factorial correlators [6, 4]. Since such a procedure was already applied to the α model with encouraging results [11], we feel that it may be rewarding to analyze the data in the more general framework of arbitrary cascade model. This should allow for a sensitive test of the relevance of the cascade models to the processes of particle production and, if they appear to be relevant, for better understanding of the structure of the cascade and of its parameters.

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⁴ This was first explored in Ref. [9].

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